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THEORY FOR MODERATELY LARGE DEFLECTIONS OF SANDWICH SHELLS WITH DISSIMILAR FACINGS

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Abstract—This paper presents the differential equations and boundary conditions for sandwich shells with moderately large rotations. The theory includes the bending resistance of the facings, transverse extension and shear deformation of the core. Approximations and simplifications are described.

NOTATION

The usual suffix notations are used. Latin suffixes represent the numbers 1, 2, and 3, while Greek suffixes represent only 1 and 2. Repeated suffixes imply summation unless enclosed by parentheses.

The arrow ($\vec{\ }$) over a symbol denotes a vector and the caret ($\hat{\ }$) denotes a unit vector. The vertical line (|) is used here to denote covariant differentiation with respect to the undeformed middle surface. A comma (,) denotes partial differentiation.

A prefix $\underline{n} = \underline{0}$ or $\underline{1}$ refers to the upper or lower facing; the prefix is underlined to avoid confusion with a suffix. Upper and lower signs, e.g. $\pm$, refer to the upper and lower facings, respectively.

Where a distinction is necessary, capital letters refer to the deformed shell while lower-case letters refer to the undeformed shell.

An element of the deformed shell, the convected coordinate lines and stress resultants are shown in Fig. 1.

The list of notations follows:

L	a characteristic length of the middle surface
d	thickness of the core
${}_{\underline{n}}d$	thickness of a facing ($\underline{n} = \underline{0}$ or $\underline{1}$)
λ	$d/2L$
$\bar{\lambda}$	$2\lambda + {}_{\underline{0}}\lambda/2 + {}_{\underline{1}}\lambda/2$
${}_{\underline{n}}\lambda$	${}_{\underline{n}}d/L$
$\tilde{\lambda}$	${}_{\underline{0}}\lambda - {}_{\underline{1}}\lambda$
θ^α	dimensionless surface coordinate
θ^3	$\dfrac{x^3}{\lambda L}$; x^3 = length along the normal to the middle surface
$\vec{a}_\alpha$ (or $\vec{A}_\alpha$)	$\vec{R}_{,\alpha}$ at the undeformed (deformed) middle surface; the position vector is $L\vec{R}$
$\vec{a}_3$ (or $\vec{A}_3$)	$\vec{R}_{,3}$
$a_{\alpha\beta}$	$\vec{a}_\alpha \cdot \vec{a}_\beta$
a	determinant $\lvert a_{\alpha\beta}\rvert$
$b_{\alpha\beta}$ (or $B_{\alpha\beta}$)	$\vec{a}_3 \cdot \vec{a}_{\alpha,\beta}$
k (or K)	Gaussian curvature
$\varepsilon_{\alpha\beta}$	$(\sqrt{a})e_{\alpha\beta}$; $e_{\alpha\beta}$ = permutation symbol
$\vec{c}$	$c_\alpha\vec{A}^\alpha$ = unit tangent to the edge
$\vec{u}$	$u_\alpha\vec{A}^\alpha$ = unit normal to the edge in the surface
g_{ij}	metric tensor associated with coordinates θ^i
g	determinant $\lvert g_{ij}\rvert$
${}_{\underline{n}}\vec{P}$	$\dfrac{1}{L^2}({}_{\underline{n}}p^\beta\,{}_{\underline{n}}\vec{A}_\beta + {}_{\underline{n}}p^3\,{}_{\underline{n}}\vec{A}_3)$, external force per unit area
${}_{\underline{n}}\vec{N}^\alpha$	$(\sqrt{a})({}_{\underline{n}}n^{\alpha\gamma}\,{}_{\underline{n}}\vec{A}_\gamma + {}_{\underline{n}}q^\alpha\,{}_{\underline{n}}\vec{A}_3)$ force resultant per unit of the θ^β coordinate line ($\beta \neq \alpha$)
${}_{\underline{n}}\vec{M}^\alpha$	$(\sqrt{a})L\varepsilon_{\gamma\rho}\,{}_{\underline{n}}m^{\alpha\gamma}\,{}_{\underline{n}}\vec{A}^\rho$, resultant couple per unit of the θ^β coordinate line ($\beta \neq \alpha$)
${}_{\underline{n}}\vec{V}$	${}_{\underline{n}}v_i\vec{a}^i$, interface displacement

${}_{\underline{0}}\vec{V} - {}_{\underline{1}}\vec{V}$	$2w_i\vec{a}^i$, relative displacement of the interfaces
${}_{\underline{0}}\vec{V} + {}_{\underline{1}}\vec{V}$	$2\bar{w}_i\vec{a}^i$, average displacement of the interfaces
$\bar{p}^i$	${}_{\underline{0}}p^i + {}_{\underline{1}}p^i$
p^3	${}_{\underline{0}}p^3 - {}_{\underline{1}}p^3$
$\tilde{p}^\gamma$	${}_{\underline{0}}p^\gamma - c\,{}_{\underline{1}}p^\gamma$
$\bar{c}^\alpha$	${}_{\underline{0}}\lambda\,{}_{\underline{0}}p^\alpha - {}_{\underline{1}}\lambda\,{}_{\underline{1}}p^\alpha$
c^α	$\lambda({}_{\underline{0}}\lambda\,{}_{\underline{0}}p^\alpha + {}_{\underline{1}}\lambda\,{}_{\underline{1}}p^\alpha)$
$\bar{n}^{\alpha\beta}$	${}_{\underline{0}}n^{\alpha\beta} + {}_{\underline{1}}n^{\alpha\beta}$
$\tilde{n}^{\alpha\beta}$	$\lambda({}_{\underline{0}}n^{\alpha\beta} - c\,{}_{\underline{1}}n^{\alpha\beta})$
$\bar{m}^{\alpha\beta}$	${}_{\underline{0}}m^{\alpha\beta} + {}_{\underline{1}}m^{\alpha\beta}$
$m^{\alpha\beta}$	$\lambda({}_{\underline{0}}m^{\alpha\beta} - {}_{\underline{1}}m^{\alpha\beta})$
$\bar{q}^\alpha$	${}_{\underline{0}}q^\alpha + {}_{\underline{1}}q^\alpha = \bar{m}^{\gamma\alpha}\|_\gamma + \frac{1}{2}\bar{c}^\alpha + \frac{{}_{\underline{0}}\hat{\lambda} + {}_{\underline{1}}\lambda}{4\lambda}\bar{s}^\alpha$
q^α	$\lambda({}_{\underline{0}}q^\alpha - {}_{\underline{1}}q^\alpha) = m^{\gamma\alpha}\|_\gamma + \frac{1}{2}c^\alpha + \frac{\tilde{\lambda}}{4}\bar{s}^\alpha$
${}_{\underline{n}}B_{\alpha\beta}$	${}_{\underline{n}}\vec{A}_3 \cdot \vec{A}_{\alpha\cdot\beta}$
$\bar{B}_{\alpha\beta}$	$\frac{1}{2}({}_{\underline{0}}B_{\alpha\beta} + {}_{\underline{1}}B_{\alpha\beta}) = b_{\alpha\beta} + \frac{1}{L^2}\bar{\kappa}_{\alpha\beta}$
$B_{\alpha\beta}$	$\frac{1}{2}({}_{\underline{0}}B_{\alpha\beta} - {}_{\underline{1}}B_{\alpha\beta}) = \frac{1}{L^2}\kappa_{\alpha\beta}$
$\bar{\kappa}_{\alpha\beta}$	$\frac{1}{2}({}_{\underline{0}}\kappa_{\alpha\beta} + {}_{\underline{1}}\kappa_{\alpha\beta}) = L^2(\bar{B}_{\alpha\beta} - b_{\alpha\beta}) = \frac{1}{2}(\bar{\omega}_{3\alpha}\|_\beta + \bar{\omega}_{3\beta}\|_\alpha)$
$\kappa_{\alpha\beta}$	$\frac{1}{2}({}_{\underline{0}}\kappa_{\alpha\beta} - {}_{\underline{1}}\kappa_{\alpha\beta}) = L^2 B_{\alpha\beta} = \frac{1}{2}(\omega_{3\alpha}\|_\beta + \omega_{3\beta}\|_\alpha)$
$\bar{\omega}_{3\alpha}$	$\frac{1}{2}({}_{\underline{0}}\omega_{3\alpha} + {}_{\underline{1}}\omega_{3\alpha}) = L(\bar{w}_3\|_\alpha + \underline{b^\lambda_\alpha \bar{w}_\lambda})$
$\omega_{3\alpha}$	$\frac{1}{2}({}_{\underline{0}}\omega_{3\alpha} - {}_{\underline{1}}\omega_{3\alpha}) = L(w_3\|_\alpha + \underline{b^\lambda_\alpha w_\lambda})$
${}_{\underline{n}}\gamma_{\alpha\beta}$	strain of the middle surface of a facing
$\bar{\gamma}_{\alpha\beta}$	$({}_{\underline{0}}\gamma_{\alpha\beta} + {}_{\underline{1}}\gamma_{\alpha\beta}/c)$
$\tilde{\gamma}_{\alpha\beta}$	$\tilde{\lambda}({}_{\underline{0}}\gamma_{\alpha\beta} - {}_{\underline{1}}\gamma_{\alpha\beta})$
τ^{ij}	stress tensor
$\bar{s}^\alpha$	$\frac{\lambda L}{(\sqrt{a})}\int_{-1}^{+1}(\sqrt{g})\tau^{3\alpha}\,\mathrm{d}\theta^3$
σ^{33}	$L^2\lambda^3({}_{\underline{0}}\tau^{33} + {}_{\underline{1}}\tau^{33})$, ${}_{\underline{n}}\tau^{33}$ denotes the transverse normal stress on the interface
Φ	an invariant stress function; c.f. equation (39)
$\bar{F}^{\alpha\beta}$	see equation (40)
$\bar{P}$	$LB\left(\bar{p}^3 + \frac{1}{2}c^\alpha\|_\alpha + \frac{\check{\lambda}}{1+c}\tilde{p}^\alpha\|_\alpha\right) - \frac{(1-\eta)}{2L\check{\lambda}\,{}_{\underline{0}}\lambda\,{}_{\underline{0}}\mu}\tilde{p}^\alpha\|^\beta_{\alpha\beta}$
χ	an invariant function; c.f. equation (54)
$\tilde{P}$	see equation (53)
A	$\frac{(1-\eta)G}{\lambda\,{}_{\underline{0}}\lambda\,{}_{\underline{0}}\mu}\left[\frac{1+c}{4} + \frac{3c\check{\lambda}^2}{({}_{\underline{1}}\lambda^2 + c\,{}_{\underline{0}}\lambda^2)}\right]$
B	$\frac{3c(1+c)(1-\eta)^2 G}{2\lambda\,{}_{\underline{0}}\lambda^2\,{}_{\underline{0}}\mu^2({}_{\underline{1}}\lambda^2 + c\,{}_{\underline{0}}\lambda^2)L^2}$
C	$\frac{4\lambda\,{}_{\underline{0}}\lambda\,{}_{\underline{0}}\mu}{(1+c)(1-\eta)G}$
$\bar{X}_i, X_i$	physical (not tensorial) components of edge forces
$\bar{H}_\alpha, H_\alpha$	physical (not tensorial) components of edge couples
$\mathscr{S}$	edge shear force on the core
$\bar{\Delta}_i, \Delta_i$	physical (not tensorial) components of edge displacements
η	Poisson's ratio of both facings
${}_{\underline{n}}\mu$	shear modulus of a facing
G^α	transverse shear modulus of an orthotropic core; $D^{\alpha\beta} = a^{\alpha\beta}G^{(\beta)}$ and $C_{\alpha\beta} = a_{\alpha\beta}/G^{(\alpha)}$

G	transverse shear modulus of an isotropic core
E	Young's modulus for transverse extension of the core
$\bar{B}^{\alpha\beta\gamma\eta}$	stiffness tensor; c.f. equations (10), (11), and (15)
$\bar{C}_{\alpha\beta\gamma\eta}$	flexibility tensor $\bar{C}_{\alpha\beta\mu\delta}\bar{B}^{\alpha\beta\gamma\eta} = \frac{1}{2}(\delta^{\gamma}_{\mu}\delta^{\eta}_{\delta}+\delta^{\gamma}_{\delta}\delta^{\eta}_{\mu})$
$D^{\alpha\beta}$	shear-stiffness tensor of the core
$C_{\alpha\beta}$	shear-flexibility tensor of the core
c	$\dfrac{{}_{\underline{0}}\lambda_{\underline{0}}\mu}{{}_{\underline{1}}\lambda_{\underline{1}}\mu}$

INTRODUCTION

In a previous paper [1] the author presented a theory of sandwich plates with dissimilar facings and in another [2], a theory for thin sandwich shells. Here, the ideas from both are incorporated in a theory for shells with dissimilar facings. The initial curvature of the shell necessitates certain approximations not needed in the treatment of plates [1] and the dissimilarity of the facings necessitates some extension of the theory for shells [2].

Although the present theory is neither as simple nor as precise as that for plates [1], it extends the ideas of the latter to shells and includes the previous formulations [1, 2] as special cases.

BASIC ASSUMPTIONS

The following analysis is concerned with small strains and moderate rotations of thin shells. It is a basic assumption that

$$2\gamma_{\alpha\beta} = A_{\alpha\beta}-a_{\alpha\beta} \ll \sqrt{[a_{(\alpha\alpha)}a_{(\beta\beta)}]}.$$

This means that dynamical equations account for any changes in the direction of lines and surfaces, but not their deformations; for example, it is immaterial whether the stress is measured per unit of deformed or undeformed area, but essential that it is referred to the deformed orientation.

Since the components of the metric tensors, $A_{\alpha\beta}$ and $a_{\alpha\beta}$, differ by terms of the order of the strain components, a covariant derivative with respect to a deformed surface is replaced by the derivative with respect to the undeformed surface.

It is assumed that rotations about a normal are small of the same order-of-magnitude as the surface strains, i.e. $\omega_{\alpha\beta} = O(\gamma_{\alpha\beta})$. This assumption appears to be valid in most structural applications. Exceptions are a long helicoidal shell [3, 4] and a cylindrical tube [5]. The latter are unusual among shell-structures because both can suffer large rotations about a normal without appreciable extension of the middle surface and the attendant membrane force.

Since the composite shell is thin, no distinctions are made between the differential geometries of the various undeformed surfaces, i.e.

$${}_{\underline{0}}a_{\alpha\beta} \doteq {}_{\underline{1}}a_{\alpha\beta} \doteq a_{\alpha\beta} \quad \text{and} \quad {}_{\underline{0}}b_{\alpha\beta} \doteq {}_{\underline{1}}b_{\alpha\beta} \doteq b_{\alpha\beta}.$$

KINEMATIC VARIABLES

According to the Kirchhoff–Love theory the configuration of a facing is defined by the displacement of the interface, ${}_{\underline{n}}\vec{V}(\theta^1, \theta^2) = {}_{\underline{n}}v_i\vec{a}^i$. However, since we are concerned with the behavior of the composite shell, it is more meaningful to employ certain linear combinations, namely, the mean displacement

$$\bar{\vec{W}} = {}_{\underline{0}}\vec{V}+{}_{\underline{1}}\vec{V} = 2\bar{w}_i\vec{a}^i \tag{1}$$

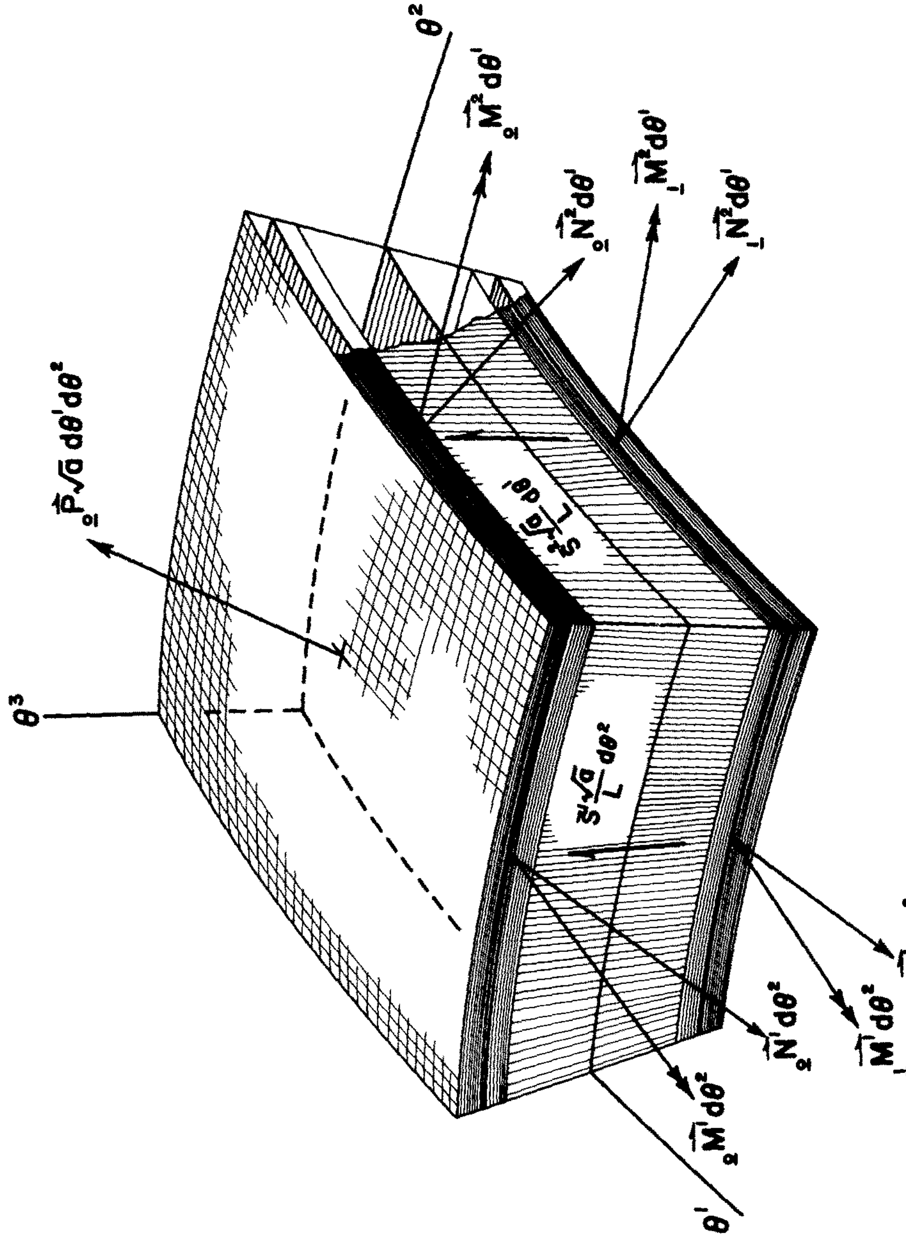

FIG. 1. Element of a deformed sandwich shell.

and the relative displacement

$$\vec{W} = {}_{\underline{0}}\vec{V} - {}_{\underline{1}}\vec{V} = 2w_i a^i. \tag{2}$$

The latter describes the deformation of the core and depends on the properties of the core and the tractions exerted upon the core by the facings.

BEHAVIOR OF THE CORE

A previous paper [6] gives a general analysis of the weak core. Here we require the approximation appropriate to a thin core ($\lambda \ll 1$); we assume that the geometric properties of the undeformed interfaces are the same. Then the appropriate versions of the core relations are

$$w_3 = \frac{L}{2E}\sigma^{33} - \frac{d}{4L^4}a^{\alpha\beta}\bar{\omega}_{3\alpha}\bar{\omega}_{3\beta} \tag{3}$$

$$w_\alpha = -\lambda(\bar{w}_3|_\alpha + b^\beta_\alpha \bar{w}_\beta) + \frac{1}{2L}C_{\alpha\beta}\bar{s}^\beta - \frac{\lambda^2}{6EL}\bar{s}^\beta|_{\beta\alpha} \tag{4a}$$

or

$$\bar{s}^\beta - \frac{\lambda^2}{3E}D^{\alpha\beta}\bar{s}^\gamma|_{\gamma\alpha} = 2LD^{\alpha\beta}w_\alpha + dD^{\alpha\beta}(\bar{w}_3|_\alpha + b^\gamma_\beta \bar{w}_\gamma) \tag{4b}$$

$D^{\alpha\beta}$ and $C_{\alpha\beta}$ are the shear stiffness and flexibility tensors, σ^{33} and $\bar{s}^\alpha$ are proportional to the transverse normal stress and shear resultant and $\bar{\omega}_{3\alpha}$ represents a gross rotation [7].

EQUILIBRIUM OF THE FACINGS

The equations [2] of equilibrium for a facing are

$${}_{\underline{n}}p^\lambda \mp \frac{1}{2\lambda}\bar{s}^\lambda + {}_{\underline{n}}n^{\alpha\lambda}|_\alpha - \underline{{}_{\underline{n}}q^\alpha {}_{\underline{n}}B^\lambda_\alpha} = 0 \tag{5}$$

$${}_{\underline{n}}p^3 \mp \frac{L^2}{2\lambda}\sigma^{33} + {}_{\underline{n}}n^{\alpha\beta}{}_{\underline{n}}B_{\alpha\beta} + {}_{\underline{n}}q^\alpha|_\alpha + \tfrac{1}{2}\bar{s}^\alpha|_\alpha = 0 \tag{6}$$

$${}_{\underline{n}}m^{\alpha\beta}|_\alpha - {}_{\underline{n}}q^\beta + \frac{{}_{\underline{n}}\lambda}{4\lambda}\bar{s}^\beta \pm \frac{{}_{\underline{n}}\lambda}{2}{}_{\underline{n}}p^\beta = 0. \tag{7}$$

The terms containing σ^{33} and $\bar{s}^\alpha$ represent the tractions exerted by the core. Notice that ${}_{\underline{n}}B_{\alpha\beta}$ denotes the second fundamental tensor of the *deformed* surface and that ${}_{\underline{n}}n^{\alpha\beta}$ and ${}_{\underline{n}}m^{\alpha\beta}$ are the force and couples with respect to the middle surface of the facing*.

The underlined term of (5) is often neglected [8, 9]. In the case of a sandwich shell it is less significant because the individual facing will usually carry a small portion of the total transverse shear resultant (e.g. ${}_{\underline{n}}q^\alpha < \bar{s}^\alpha$). Consequently, it will suffice to set ${}_{\underline{0}}B_{\alpha\beta} = {}_{\underline{1}}B_{\alpha\beta} = \bar{B}_{\alpha\beta}$ in the underlined term only.

If we use (7) to eliminate ${}_{\underline{n}}q^\alpha$ from (5) and (6), the equilibrium equations are

$${}_{\underline{n}}p^\beta\left(\delta^\lambda_\beta \mp \underline{\frac{{}_{\underline{n}}\lambda}{2}\bar{B}^\lambda_\beta}\right) \mp \frac{1}{2\lambda}\bar{s}^\beta\left(\delta^\lambda_\beta \pm \underline{\frac{{}_{\underline{n}}\lambda}{2}\bar{B}^\lambda_\beta}\right) + {}_{\underline{n}}n^{\alpha\lambda}|_\alpha - \underline{\bar{B}^\lambda_\beta {}_{\underline{n}}m^{\alpha\beta}|_\alpha} = 0 \tag{8}$$

* The author's previous work [2, 6] dealt with stress resultants acting at the interface. Note too that the stress-resultant tensors, ${}_{\underline{n}}n^{\alpha\beta}$, ${}_{\underline{n}}q^\alpha$, ${}_{\underline{n}}m^{\alpha\beta}$ differ with their counterparts in [8] by the factor L, but agree with the notations of [6].

$$ {}_{\underline{n}}p^3 \mp \frac{L^2}{2\lambda}\sigma^{33} + \tfrac{1}{2}\left(1+\frac{{}_{\underline{n}}\lambda}{2\lambda}\right)\bar{s}^{\lambda}|_{\lambda} + {}_{\underline{n}}n^{\alpha\beta}{}_{\underline{n}}B_{\alpha\beta} + {}_{\underline{n}}m^{\alpha\beta}|_{\alpha\beta} \pm \frac{{}_{\underline{n}}\lambda}{2}{}_{\underline{n}}p^{\lambda}|_{\lambda} = 0. \tag{9} $$

BENDING AND STRETCHING OF A FACING

The Kirchhoff–Love hypothesis is employed to obtain the constitutive relations [10]:

$$ {}_{\underline{n}}n^{\alpha\beta} = {}_{\underline{n}}\lambda_{\underline{n}}B^{\alpha\beta\gamma\eta}{}_{\underline{n}}\gamma_{\gamma\eta}, \qquad {}_{\underline{n}}m^{\alpha\beta} = -\frac{{}_{\underline{n}}\lambda^3}{12}{}_{\underline{n}}B^{\alpha\beta\gamma\eta}{}_{\underline{n}}\kappa_{\gamma\eta} \tag{10, 11} $$

where

$$ {}_{\underline{n}}\gamma_{\alpha\beta} = \frac{L}{2}\left({}_{\underline{n}}v_{\alpha}|_{\beta} + {}_{\underline{n}}v_{\beta}|_{\alpha} - 2b_{\alpha\beta\underline{n}}v_3 + \frac{1}{L^3}{}_{\underline{n}}\omega_{3\alpha\underline{n}}\omega_{\alpha\beta}\right) \mp \frac{{}_{\underline{n}}\lambda}{2}{}_{\underline{n}}\kappa_{\alpha\beta} \tag{12} $$

$$ {}_{\underline{n}}\kappa_{\alpha\beta} = \frac{L}{2}[({}_{\underline{n}}v_3|_{\alpha\beta} + {}_{\underline{n}}v_3|_{\beta\alpha}) + \underline{(b^{\lambda}_{\alpha\underline{n}}v_{\lambda})|_{\beta}} + \underline{(b^{\lambda}_{\beta\underline{n}}v_{\lambda})|_{\alpha}}] \tag{13} $$

${}_{\underline{n}}\omega_{3\alpha}$ is associated with rotations about a tangent [7]:

$$ {}_{\underline{n}}\omega_{3\alpha} = L({}_{\underline{n}}v_3|_{\alpha} + \underline{b^{\lambda}_{\alpha\underline{n}}v_{\lambda}}). \tag{14} $$

For practicality we have neglected some terms, specifically, a quadratic term in (12) and a linear term in (13) which involve the rotation (${}_{\underline{n}}\omega_{\alpha\beta}$) about the normal and an asymmetric term of (10). These approximations [10] are justified when the rotation about the normal is sufficiently small as it is usually. When we neglect the underlined terms in the equilibrium equations (5) and (8), then, to be consistent, we will neglect the tangential displacements in (13) [8] when (13) is used in (11).

We suppose that the elastic properties of the facings are similar, but the stiffness and thickness may differ. Accordingly,

$$ {}_{\underline{n}}B^{\alpha\beta\gamma\eta} = {}_{\underline{n}}\mu\bar{B}^{\alpha\beta\gamma\eta} \tag{15} $$

SANDWICH-SHELL VARIABLES

In the manner of equations (1) and (2), we define rotations $\bar{\omega}_{3\alpha}$ and $\omega_{3\alpha}$, changes-of-curvature $\bar{\kappa}_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ and curvatures $\bar{B}_{\alpha\beta}$ and $B_{\alpha\beta}$. Likewise we combine the loads, stress-resultants and strain components of the individual facings to form new variables, $\bar{q}^{\alpha}$, $\bar{n}^{\alpha\beta}$, $\bar{m}^{\alpha\beta}$, $\bar{p}^i$, $\bar{c}^{\alpha}$, $\bar{\gamma}_{\alpha\beta}$, q^{α}, $\tilde{n}^{\alpha\beta}$, $m^{\alpha\beta}$, p^3, $\tilde{p}^{\alpha}$, c^{α} and $\tilde{\gamma}_{\alpha\beta}$. All definitions are given under the heading of "Notations". Each barred ($^-$) quantity is associated with the gross behavior, for example, $\bar{n}^{\alpha\beta}$ is a net-force component on the composite.

According to (12), (2) and the definitions,

$$ \begin{aligned} \bar{\gamma}_{\alpha\beta} = \frac{L}{2c}\Big[&(1+c)(\bar{w}_{\alpha}|_{\beta} + \bar{w}_{\beta}|_{\alpha}) - (1-c)(w_{\alpha}|_{\beta} + w_{\beta}|_{\alpha}) \\ &-2(1+c)b_{\alpha\beta}\bar{w}_3 + 2(1-c)b_{\alpha\beta}w_3 + \frac{1}{L^3}(1+c)\bar{\omega}_{3\alpha}\bar{\omega}_{3\beta} \\ &+\frac{1}{L^3}(1+c)\omega_{3\alpha}\omega_{3\beta} - \frac{1}{L^3}(1-c)\omega_{3\alpha}\bar{\omega}_{3\beta} - \frac{1}{L^3}(1-c)\bar{\omega}_{3\alpha}\omega_{3\beta}\Big] \\ &-\frac{(c_{\underline{0}}\lambda - {}_{\underline{1}}\lambda)}{2c}\bar{\kappa}_{\alpha\beta} - \frac{(c_{\underline{0}}\lambda + {}_{\underline{1}}\lambda)}{2c}\kappa_{\alpha\beta} \end{aligned} \tag{16a} $$

$$\tilde{\gamma}_{\alpha\beta} = \lambda L \left(w_\alpha|_\beta + w_\beta|_\alpha - 2b_{\alpha\beta}w_3 + \frac{1}{L^3}\bar{\omega}_{3\alpha}\omega_{3\beta} + \frac{1}{L^3}\omega_{3\alpha}\bar{\omega}_{3\beta} \right)$$
$$-\frac{\lambda}{2}({}_0\lambda + {}_1\lambda)\bar{\kappa}_{\alpha\beta} - \frac{\lambda}{2}({}_0\lambda - {}_1\lambda)\kappa_{\alpha\beta}. \quad (16b)$$

$$\bar{\kappa}_{\alpha\beta} = \frac{L}{2}[(\bar{w}_3|_\alpha + \underline{b^\lambda_\alpha \bar{w}_\lambda})|_\beta + (\bar{w}_3|_\beta + \underline{b^\lambda_\beta \bar{w}_\lambda})|_\alpha] \quad (17a)$$

$$\kappa_{\alpha\beta} = \frac{L}{2}[(w_3|_\alpha + \underline{b^\lambda_\alpha w_\lambda})|_\beta + (w_3|_\beta + \underline{b^\lambda_\beta w_\lambda})|_\alpha] \quad (17b)$$

Notice that the tangential displacements $\bar{w}_\alpha$ appear in (16b) only in the nonlinear terms and in $\bar{\kappa}_{\alpha\beta}$ where they are underlined in (17a).

EQUILIBRIUM OF THE SANDWICH SHELL

Next we form linear combinations of (8) and (9) to obtain equations which are more meaningful for the composite shell. These combinations are ${}_0(8) + {}_1(8)$, ${}_0(8) - c_1(8)$, ${}_0(9) + {}_1(9)$ and ${}_0(9) - {}_1(9)$, respectively:

$$\bar{p}^\lambda + \bar{n}^{\alpha\lambda}|_\alpha - \underline{\tfrac{1}{2}\bar{B}^\lambda_\beta \bar{c}^\beta} - \underline{\frac{({}_0\lambda + {}_1\lambda)}{4\lambda}\bar{B}^\lambda_\beta \bar{s}^\beta} - \underline{\bar{B}^\lambda_\beta \bar{m}^{\alpha\beta}|_\alpha} = 0 \quad (18)$$

$$\lambda\tilde{p}^\lambda - \frac{(1+c)}{2}\bar{s}^\lambda + \tilde{n}^{\alpha\lambda}|_\alpha - \underline{\frac{\lambda}{2(1+c)}[c({}_0\lambda + {}_1\lambda)\bar{p}^\beta}$$
$$+ \underline{({}_0\lambda - c_1\lambda)\tilde{p}^\beta]\bar{B}^\lambda_\beta} - \underline{\frac{({}_0\lambda - {}_1\lambda c)}{4}\bar{B}^\lambda_\beta \bar{s}^\beta} \quad (19)$$
$$-\underline{\tfrac{1}{2}[\lambda(1-c)\bar{m}^{\alpha\beta}|_\alpha + (1+c)m^{\alpha\beta}|_\alpha]\bar{B}^\lambda_\beta} = 0$$

$$\bar{p}^3 + \frac{\bar{\lambda}}{2\lambda}\bar{s}^\lambda|_\lambda + \tfrac{1}{2}\bar{c}^\lambda|_\lambda + \bar{n}^{\alpha\beta}\bar{B}_{\alpha\beta}$$
$$+ \frac{2}{\lambda(1+c)}\tilde{n}^{\alpha\beta}B_{\alpha\beta} - \frac{(1-c)}{(1+c)}\bar{n}^{\alpha\beta}B_{\alpha\beta} + \bar{m}^{\alpha\beta}|_{\alpha\beta} = 0 \quad (20)$$

$$\lambda p^3 - L^2\sigma^{33} + \frac{{}_0\lambda - {}_1\lambda}{4}\bar{s}^\lambda|_\lambda + \tfrac{1}{2}c^\alpha|_\alpha$$
$$+ \frac{2}{(1+c)}\tilde{n}^{\alpha\beta}\bar{B}_{\alpha\beta} + \lambda\bar{n}^{\alpha\beta}B_{\alpha\beta} + \lambda\frac{(c-1)}{(c+1)}\bar{n}^{\alpha\beta}\bar{B}_{\alpha\beta} + m^{\alpha\beta}|_{\alpha\beta} = 0. \quad (21)$$

Equations (18) and (20) are the conditions for equilibrium of the net forces on the composite shell.

BENDING AND STRETCHING OF THE COMPOSITE SHELL

Similar linear combinations of (10) and (11) are:

$$\bar{n}^{\alpha\beta} = {}_0\lambda_0\mu\bar{B}^{\alpha\beta\gamma\eta}\bar{\gamma}_{\gamma\eta} \tag{22}$$

$$\tilde{n}^{\alpha\beta} = {}_0\lambda_0\mu\bar{B}^{\alpha\beta\gamma\eta}\tilde{\gamma}_{\gamma\eta} \tag{23}$$

$$\bar{m}^{\alpha\beta} = -\frac{{}_0\lambda^3{}_0\mu}{12}\bar{B}^{\alpha\beta\gamma\eta}\left[\left(1+\frac{{}_1\lambda^2}{c_0\lambda^2}\right)\bar{\kappa}_{\gamma\eta}+\left(1-\frac{{}_1\lambda^2}{c_0\lambda^2}\right)\kappa_{\gamma\eta}\right] \tag{24}$$

$$m^{\alpha\beta} = -\frac{{}_0\lambda^2{}_0\mu}{12}\lambda\bar{B}^{\alpha\beta\gamma\eta}\left[\left(1-\frac{{}_1\lambda^2}{c_0\lambda^2}\right)\bar{\kappa}_{\gamma\eta}+\left(1+\frac{{}_1\lambda^2}{c_0\lambda^2}\right)\kappa_{\gamma\eta}\right]. \tag{25}$$

Equations (22), (23), (24), and (25) together with (16a, b) and (17a, b) serve to express the stress resultants in terms of the displacements w_i and $\bar{w}_i$. Then the six equilibrium equations (18–21) can be expressed in terms of the variables w_i, $\bar{w}_i$, σ^{33} and $\bar{s}^\alpha$. The relative displacements w_i can be eliminated easily by means of (3) and (4a); then the six equations (18)–(21) are expressed in terms of the six variables, $\bar{w}_i$, σ^{33} and $\bar{s}^\alpha$.

Usually the coefficient $\lambda^2 D^{\alpha\beta}/3E$ is small enough that the second term of (4b) is negligible. (The importance of this term and w_3 have been discussed by Reissner [11].) Then equations (4) serve as well to eliminate the transverse shear resultants $\bar{s}^\alpha$ so that the six equations (18)–(21) can be expressed in terms of the six displacement components, $\bar{w}_i$ and w_i. The underlined terms of (18) and (19) are usually neglected when the pattern of deformations is limited to a shallow region; they vanish in a membrane theory. We retain these terms but employ the simplifying approximation

$$\bar{B}_{\alpha\beta} \doteq b_{\alpha\beta}.$$

The eight equations, (18)–(25), together with (3) and (4) form the basis of the governing differential equations. Appropriate boundary conditions and some simplifications are discussed in the following.

BOUNDARY CONDITIONS

The boundary conditions can be obtained by examining the virtual work of edge tractions in the manner of previous papers [1, 2]. Here we present the results and define some essential quantities.

We introduce a mean base vector $\vec{A}_i$ and gross rotation $\bar{\bar{\phi}}$ of the edge of the composite shell:

$$\vec{A}_i = \vec{a}_i + \bar{\bar{\vec{\phi}}} \times \vec{a}_i \tag{26}$$

$$\bar{\bar{\phi}}_i = \frac{\varepsilon^{\gamma\alpha}}{L}(\bar{w}_3|_\alpha + b_\alpha^\lambda \bar{w}_\lambda)\vec{a}_\gamma. \tag{27}$$

The actual convected base vectors $\vec{G}_i(\theta^1, \theta^2, \theta^3)$ differ from $\vec{A}_i(\theta^1, \theta^2)$ by virtue of relative rotation and strain variations through the thickness. These variations lie beyond the realm of thin shell theories. Only the gross rotation and the relative rotation of the two facings enters into our theory for thin-sandwich shells.

Let $\hat{c} = c_\alpha \vec{A}^\alpha$ and $\hat{u} = u_\alpha \vec{A}^\alpha$ be the unit tangent and normal to an edge curve C of the deformed shell. Let $\bar{\vec{\delta}}$ and $2\vec{\delta}$ be the mean and relative displacements of the middle surfaces of the facings, $\bar{\vec{\eta}}$ and $\vec{\eta}$ the sum and difference of the force resultants on the facings, $\bar{\vec{m}}$ and $\vec{m}$ the sum and difference of the couple resultants and $\mathscr{S}$ the shear resultant on the core. These are expressed in terms of normal and tangential components as follows:

$$\bar{\vec{\delta}} = \bar{\Delta}_1\hat{u}+\bar{\Delta}_2\hat{c}+\bar{\Delta}_3\hat{A}_3 \tag{28a}$$

$$\vec{\delta} = \Delta_1\hat{u}+\Delta_2\hat{c}+\Delta_3\hat{A}_3 \tag{28b}$$

$$\bar{\vec{\eta}} = \bar{X}_1\hat{u}+\bar{X}_2\hat{c}+\bar{X}_3\hat{A}_3 \tag{29a}$$

$$\vec{\eta} = X_1\hat{u}+X_2\hat{c}+X_3\hat{A}_3 \tag{29b}$$

$$\bar{\vec{m}} = \bar{H}_1\hat{u}+\bar{H}_2\hat{c} \tag{30a}$$

$$\vec{m} = H_1\hat{u}+H_2\hat{c}. \tag{30b}$$

With the foregoing notations, the virtual work of the edge tractions takes the form

$$\begin{aligned}\delta\Omega = \int_C\Bigg\{&\left[\bar{X}_3+\mathscr{S}-\frac{\partial\bar{H}_1}{\partial C}\right]\bar{\Delta}_3-[\bar{H}_2]\frac{\partial\bar{\Delta}_3}{\partial U}+\left[X_3-\frac{\partial H_1}{\partial C}\right]\Delta_3\\ &-[H_2]\frac{\partial\Delta_3}{\partial U}+[\bar{X}_1+\underline{\bar{H}_1\tau-\bar{H}_2K_1}]\bar{\Delta}_1\\ &+[X_1+\underline{H_1\tau-H_2K_1}]\Delta_1+[\bar{X}_2+\underline{\bar{H}_1K_2-\bar{H}_2\tau}]\bar{\Delta}_2\\ &+[X_2+\underline{H_1K_2-H_2\tau}]\Delta_2\Bigg\}\,\mathrm{d}C.\end{aligned} \tag{31}$$

Here K_2 and K_1 are normal curvatures in the tangential and normal directions, τ is the geodesic torsion of the surface with fundamental tensors $A_{\alpha\beta}$ and $B_{\alpha\beta}$, C and U are lengths along the curve C and a normal curve, respectively.

The underlined terms of (31) stem from the tangential components of displacement which enter into the rotations. These terms are neglected in the Donnell [12] approximation.

According to (31) there are eight boundary conditions in general. There may be geometric constraints, i.e. $\bar{w}_i$, w_i, $\partial\bar{w}_3/\partial U$ and $\partial w_3/\partial U$ may be assigned, or the edge resultants (in brackets) may be assigned or linear combinations may be prescribed, for example, in the case of an elastic support.

If we retain only those nonlinear terms which involve products of forces and rotations, then the effective edge resultants are expressed in terms of previous variables as follows:

$$\lambda LX_3 = u_\alpha q^\alpha+\frac{\lambda}{L}u_\alpha\bar{n}^{\alpha\beta}(w_3|_\beta+b^\eta_\beta w_\eta) \tag{32a}$$

$$\bar{H}_1 = -c_\alpha u_\beta\bar{m}^{\alpha\beta}, \qquad \lambda H_1 = -c_\alpha u_\beta m^{\alpha\beta} \tag{32b, c}$$

$$\bar{H}_2 = u_\alpha u_\beta\bar{m}^{\alpha\beta}, \qquad \lambda H_2 = u_\alpha u_\beta m^{\alpha\beta} \tag{32d, e}$$

$$L\bar{X}_1 = u_\alpha u_\gamma\bar{n}^{\alpha\gamma} \tag{32f}$$

$$L\bar{X}_2 = c_\alpha u_\gamma\bar{n}^{\alpha\gamma} \tag{32g}$$

$$\lambda L\tilde{X}_1 = \lambda L\frac{(1+c)}{2}X_1+\lambda L\frac{(1-c)}{2}\bar{X}_1 = u_\alpha u_\gamma\tilde{n}^{\alpha\gamma} \tag{32h}$$

$$\lambda L\tilde{X}_2 = \lambda L\frac{(1+c)}{2}X_2+\lambda L\frac{(1-c)}{2}\bar{X}_2 = c_\alpha u_\gamma \bar{n}^{\alpha\gamma} \quad (32i)$$

$$L\mathscr{S} = u_\alpha \bar{s}^\alpha. \quad (32j)$$

The effective edge forces can be expressed by components in the directions of the vectors $\vec{a}_i$ by means of (26) and (27).

SIMPLIFYING APPROXIMATIONS

If we admit certain approximations of the foregoing equations, then we can eliminate the tangential displacements, $\bar{w}_\alpha$, and instead, introduce an Airy stress function for $\bar{n}^{\alpha\beta}$ and a compatibility condition to insure the existence of $\bar{w}_\alpha$. These approximations impose some limitations on the applicability of the results. Nonetheless, the resulting equations govern a wide variety of common cases including

(i) shallow shells or, equivalently, localized and wave-like deflections of non-shallow shells and

(ii) cylindrical and spherical shells with thin facings, i.e. ${}_n\lambda \ll \lambda$.

Usually the facings are thin in comparison with the core, i.e. ${}_n\lambda \ll \lambda$. Then the shear force and the couple, ${}_nq^\alpha$ and ${}_nm^{\alpha\beta}$, on a facing are small in comparison with the net shear and couple on the composite shell. Therefore, we propose to neglect the underlined terms in the equilibrium equations (5), (8), (18), and (19). In particular, equation (18) reduces to

$$\bar{p}^\lambda + \bar{n}^{\alpha\lambda}|_\alpha = 0. \quad (33)$$

In keeping with the approximation (33) we neglect the underlined terms of (13) and like terms in the changes-of-curvature $\bar{\kappa}_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ of (24) and (25)*.

To neglect the underlined terms in $\bar{\omega}_{3\alpha}$ and $\omega_{3\alpha}$ when computing the quadratic terms of (16) or the curvatures $\bar{B}_{\alpha\beta}$ and $B_{\alpha\beta}$ is another matter, for the magnitude of the non-linear terms, e.g. $\bar{\omega}_{3\alpha}\bar{\omega}_{3\beta}$ in (16) and $\bar{n}^{\alpha\beta}\bar{B}_{\alpha\beta}$ in (20), does not rest upon the relative thickness of the facings. However, when large rotations and curvature-changes occur in the analysis of a shallow shell or, equivalently, in the study of localized or wave-like deflections of short length, then the underlined terms can be neglected throughout. In such cases we have the generalized Donnell [12] approximations for flexure of the facings.

$$\bar{\omega}_{3\alpha} = L\bar{w}_3|_\alpha, \qquad \omega_{3\alpha} = Lw_3|_\alpha \quad (34a, b)$$

$$\bar{\kappa}_{\alpha\beta} = \frac{L}{2}(\bar{w}_3|_{\alpha\beta}+\bar{w}_3|_{\beta\alpha}) \quad (35a)$$

$$\kappa_{\alpha\beta} = \frac{L}{2}(w_3|_{\alpha\beta}+w_3|_{\beta\alpha}). \quad (35b)$$

The tangential displacement $\bar{w}_\alpha$ appears in the parenthetical term of (4a, b) and enters into (23) via (16b). That term is $\bar{\omega}_{3\alpha}/L$ and represents a mean rotation about a tangent. When (4a) is substituted into (16b) the contribution of the parenthetical term is $-2\lambda^2\bar{\kappa}_{\alpha\beta}$. This is usually the predominant term in (16b). Indeed, if $c = 1$ and ${}_n\lambda \ll \lambda$, then for all

* These simplifications are often introduced in analyses of shallow shells [10]. Here we can apply them to a facing without imposing such restrictions on the composite shell.

practical purposes $\tilde{n}^{\alpha\beta}$ is the net couple on the composite shell. If the flexural resistance of the composite is appreciable, then the tangential displacement $\bar{w}_\alpha$ can be neglected in (4a, b) *only* if the deformation pattern is limited to a shallow portion of the shell. In short, to neglect $\bar{w}_\gamma b^\gamma_\alpha$ in (4a, b) is to impose the Donnell approximation on the composite shell.

Koiter [10] has suggested an approximation which can be employed here to eliminate the tangential displacement from (16b) and hence from (23). It is consistent with our assumption that the rotation $({}_{\underline{n}}\omega_{\alpha\beta})$ about the normal is small, of the same order-of-magnitude as the strain $({}_{\underline{n}}\gamma_{\alpha\beta})$. Under this assumption we use the approximation:*

$$\bar{w}_\alpha|_\beta = b_{\alpha\beta}\bar{w}_3. \tag{36}$$

When we differentiate (4a) and use (36), we obtain

$$w_\alpha|_\beta = -\lambda(\bar{w}_3|_{\alpha\beta} + \underset{\sim\sim\sim}{b^\gamma_\alpha b_{\gamma\beta}\bar{w}_3 + b^\gamma_\alpha|_\beta \bar{w}_\gamma}) + \frac{1}{2L}\left(C_{\alpha\gamma}\bar{s}^\gamma - \frac{\lambda^2}{3E}\bar{s}^\gamma\Big|_{\gamma\alpha}\right)\Big|_\beta. \tag{37}$$

Following Koiter [10] we neglect only the second of the underscored terms in (37) (it is zero if the shell is cylindrical or spherical and negligible if the shell is shallow). Then,

$$w_\alpha|_\beta = -\lambda(\bar{w}_3|_{\alpha\beta} + \underset{\sim\sim\sim}{b^\gamma_\alpha b_{\gamma\beta}\bar{w}_3}) + \frac{1}{2L}\left(C_{\alpha\gamma}\bar{s}^\gamma - \frac{\lambda^2}{3E}\bar{s}^\gamma\Big|_{\gamma\alpha}\right)\Big|_\beta. \tag{38}$$

If (38), (34a, b) and (35a, b) are used in (16b) the tangential displacement $\bar{w}_\alpha$ is eliminated from (16b) and (23).

The approximations (33), (34), (35) and (38) are employed in the sequel.

STRESS FUNCTION

If the Gaussian curvature (k) is constant† the general solution of (33) is [13]

$$\bar{n}^{\alpha\beta} = \varepsilon^{\alpha\eta}\varepsilon^{\beta\gamma}\Phi|_{\gamma\eta} + \underset{\sim\sim\sim}{a^{\alpha\beta}k}\Phi - \bar{F}^{\alpha\beta} \tag{39}$$

where $\bar{F}^{\alpha\beta}$ is a symmetric particular integral of

$$\bar{F}^{\alpha\beta}|_\alpha = \bar{p}^\beta. \tag{40}$$

Then the two equilibrium equations (33) are eliminated from further consideration and the tensions are given by (39), but only if the Gaussian curvature is constant, e.g. cylinders, spheres. In the case of shallow shells [8] the Gaussian curvature is negligibly small in (39).

A COMPATIBILITY CONDITION

It is clear from (39) and (22) that Φ is related to the strain components $\bar{\gamma}_{\alpha\beta}$ and so to $\bar{w}_\alpha$. By introducing Φ we hope that calculation of the displacements $\bar{w}_\alpha$ will be unnecessary

* This approximation is used only to compute a mean change-of-curvature in (16b). It follows when the linear versions of ${}_{\underline{n}}\gamma_{\alpha\beta}$ and ${}_n\omega_{\alpha\beta}$ are set to zero.

† Strictly speaking, this should refer to the Gaussian curvature K of the deformed surface rather than the curvature k of the undeformed surface. However, the underlined term of (39) is needed only when the deformed portion does not qualify as a shallow surface, (see [8], p. 400). Moreover, any change in the Gaussian curvature is accompanied by stretching of the surface, the change being proportional to the strain components and their derivatives. Accordingly, it is reasonable to assume that $(K-k) \ll k$ when the deformed surface is not shallow, i.e. when $K \doteq k$ is not negligible in (39).

and that Φ can be chosen to insure continuity of the deformed shell. To this end we require a compatibility condition.

A compatibility equation for each facing is obtained from the Gauss equations of the deformed and undeformed middle surfaces [1]. For small strains the equation assumes the form

$$\varepsilon^{\alpha\beta}\varepsilon^{\rho\gamma}{}_{\underline{n}}\gamma_{\alpha\gamma}|_{\beta\rho} = \frac{L^2}{2}\varepsilon^{\alpha\beta}\varepsilon^{\rho\gamma}({}_{\underline{n}}B_{\alpha\rho\underline{n}}B_{\beta\gamma} - b_{\alpha\rho}b_{\beta\gamma}). \tag{41}$$

Following the derivation of (22) we form the combination, ${}_{\underline{0}}(41)+{}_{\underline{1}}(41)/c$, and use the definitions of $\bar{\gamma}_{\alpha\beta}$, $\bar{B}_{\alpha\beta}$ and $B_{\alpha\beta}$ to obtain

$$\begin{aligned} {}_{\underline{0}}\lambda_{\underline{0}}\mu\varepsilon^{\alpha\beta}\varepsilon^{\rho\gamma}\bar{\gamma}_{\alpha\gamma}|_{\beta\rho} &= {}_{\underline{0}}\lambda_{\underline{0}}\mu\varepsilon^{\alpha\beta}\varepsilon^{\rho\gamma}\Bigg[\left(\frac{1+c}{c}\right)b_{\alpha\rho}\bar{\kappa}_{\beta\gamma} \\ &\quad -\left(\frac{1+c}{c}\right)b_{\alpha\rho}\kappa_{\beta\gamma} \\ &\quad +\frac{1}{2L^2}\left(\frac{1+c}{c}\right)\bar{\kappa}_{\alpha\rho}\bar{\kappa}_{\beta\gamma}+\frac{1}{2L^2}\left(\frac{1+c}{c}\right)\kappa_{\alpha\rho}\kappa_{\beta\gamma} \\ &\quad -\frac{1}{L^2}\left(\frac{1-c}{c}\right)\bar{\kappa}_{\alpha\rho}\kappa_{\beta\gamma}\Bigg] \\ &\equiv \mathscr{F}. \end{aligned} \tag{42}$$

where $\mathscr{F}$ is merely an abbreviation for the right side of (42).

If $\bar{C}_{\alpha\beta\mu\delta}/{}_{\underline{0}}\lambda_{\underline{0}}\mu$ denotes the flexibility tensor then, according to (39)

$$\bar{\gamma}_{\alpha\beta} = \frac{1}{{}_{\underline{0}}\lambda_{\underline{0}}\mu}\bar{C}_{\alpha\beta\gamma\eta}\bar{n}^{\gamma\eta} \tag{43a}$$

$$= \frac{1}{{}_{\underline{0}}\lambda_{\underline{0}}\mu}\bar{C}_{\alpha\beta\gamma\eta}[\varepsilon^{\gamma\mu}\varepsilon^{\eta\delta}\Phi|_{\delta\mu}+\underset{\sim\sim\sim}{a^{\gamma\eta}}k\Phi-\bar{F}^{\gamma\eta}] \tag{43b}$$

If we limit our attention to homogeneous facings such that $\bar{C}_{\alpha\beta\gamma\eta}|_\mu = 0$ then from (42) and (43b) we obtain

$$\varepsilon^{\alpha\beta}\varepsilon^{\rho\gamma}\{\bar{C}_{\alpha\gamma\mu\eta}[\varepsilon^{\mu\delta}\varepsilon^{\eta\xi}\Phi|_{\xi\delta\beta\rho}+\underset{\sim\sim\sim\sim}{a^{\mu\eta}k\Phi|_{\beta\rho}}-\bar{F}^{\mu\eta}|_{\beta\rho}]\} = \mathscr{F}. \tag{44}$$

If the facings are isotropic, (44) takes the form

$$\begin{aligned} &\Phi|_{\beta\alpha}^{\cdot\cdot\alpha\beta}-\frac{\eta}{1+\eta}(\Phi|_\alpha^\alpha)|_\beta^\beta+\underset{\sim\sim\sim\sim}{k\frac{1-\eta}{1+\eta}\Phi|_\alpha^\alpha} \\ &+\varepsilon^{\alpha\beta}\varepsilon^{\mu\delta}\bar{F}_{\alpha\delta}|_{\beta\mu}+\frac{\eta}{1+\eta}\bar{F}_\eta^\eta|_\beta^\beta = -2\mathscr{F}. \end{aligned} \tag{45}$$

The invariant $\mathscr{F}$ is readily expressed in terms of the transverse displacements $\bar{w}_3$ and w_3 by means of (42) and (35).

FORMULATION WITH THE STRESS FUNCTIONS

The introduction of the stress function Φ serves to eliminate the equilibrium conditions (33) in favor of the compatibility condition (44) or (45) which insures the continuity of $\bar{w}_\alpha$. It remains to introduce the simplifying approximations of (34), (35) and (38) into (16b), (23), (24) and (25) and, then to rewrite the equilibrium equations (19), (20) and (21) in terms of $\bar{w}_3$, w_3, s^α and Φ. Substituting (34), (35) and (38) into (16b) and then (16b) into (23) we obtain

$$\begin{aligned}\tilde{n}^{\alpha\beta} = {}_{\underline{0}}\lambda_{\underline{0}}\mu\lambda\bar{B}^{\alpha\beta\gamma\eta}\Big[&-\bar{\lambda}L\bar{w}_3|_{\gamma\eta}-\frac{\bar{\lambda}L}{2}w_3|_{\gamma\eta}-\underset{\sim\sim\sim\sim\sim}{2\lambda Lb^{\mu}_{\gamma}b_{\mu\eta}\bar{w}_3}-2Lb_{\gamma\eta}w_3\\ &+(C_{\gamma\mu}\bar{s}^{\mu})\|_{\eta}-\frac{\lambda^2}{3E}\bar{s}^{\mu}|_{\mu\gamma\eta}+2\bar{w}_3|_{\gamma}w_3|_{\eta}\Big].\end{aligned} \tag{46}$$

Substituting (35) into (24) and (25) we have

$$\bar{m}^{\alpha\beta} = -\frac{{}_{\underline{0}}\lambda^3_{\underline{0}}\mu L}{12}\bar{B}^{\alpha\beta\gamma\eta}\left[\left(1+\frac{{}_1\lambda^2}{c_{\underline{0}}\lambda^2}\right)\bar{w}_3|_{\gamma\eta}+\left(1-\frac{{}_1\lambda^2}{c_{\underline{0}}\lambda^2}\right)w_3|_{\gamma\eta}\right] \tag{47}$$

$$m^{\alpha\beta} = -\frac{{}_{\underline{0}}\lambda^3_{\underline{0}}\mu\lambda L}{12}\bar{B}^{\alpha\beta\gamma\eta}\left[\left(1-\frac{{}_1\lambda^2}{c_{\underline{0}}\lambda^2}\right)\bar{w}_3|_{\gamma\eta}+\left(1+\frac{{}_1\lambda^2}{c_{\underline{0}}\lambda^2}\right)w_3|_{\gamma\eta}\right]. \tag{48}$$

Equations (39), (46), (47) and (48) may be substituted into the equilibrium equations (19) (less the underlined terms), (20) and (21). These equations (19), (20) and (21), together with (44) or (45) constitute a system of five equations in the five variables Φ, $\bar{w}_3$, w_3, $\bar{s}^1$ and $\bar{s}^2$.

FORMULATION FOR SHALLOW ISOTROPIC SHELLS

When the shell is shallow, we are justified in neglecting the underlined term of (38) as well as those previously dropped*. Also, we can neglect the underlined term of (39) and (45)†. Furthermore, the order of covariant differentiation is irrelevant†.

If the facings and core are isotropic, equations (19) can be differentiated and summed to obtain one equation in the invariant $\bar{s}^\lambda|_\lambda$. After eliminating $\tilde{n}^{\alpha\beta}$ with (46) and deleting the underlined terms, we obtain

$$\begin{aligned}\lambda\tilde{p}^{\lambda}|_{\lambda}-\frac{(1+c)}{2}\bar{s}^{\lambda}|_{\lambda}+\frac{2_{\underline{0}}\lambda_{\underline{0}}\mu\lambda}{1-\eta}\Big[&-\bar{\lambda}L\bar{w}_3|^{\alpha\beta}_{\alpha\beta}\\ &-\frac{\bar{\lambda}L}{2}w_3|^{\alpha\beta}_{\alpha\beta}-2L(1-\eta)(b^{\alpha}_{\beta}w_3)|^{\beta}_{\alpha}\\ &-2L\eta(b^{\alpha}_{\alpha}w_3)|^{\beta}_{\beta}+\frac{1}{G}\bar{s}^{\alpha}|^{\beta}_{\alpha\beta}-\frac{\lambda^2}{3E}\bar{s}^{\mu}|^{\alpha\beta}_{\mu\alpha\beta}\\ &+2(1-\eta)(\bar{w}_3|^{\alpha}w_3|_{\beta})|^{\beta}_{\alpha}+2\eta(\bar{w}_3|_{\gamma}w_3|^{\gamma})|^{\alpha}_{\alpha}\Big]=0.\end{aligned} \tag{49}$$

Thus the two equations (19) are replaced by one, (49), and the two variables $\bar{s}^\lambda$ by the invariant $\bar{s}^\lambda|_\lambda$.

* This means that the generalized Donnell-type approximation is enforced throughout.

† These approximations are discussed by Green and Zerna [8] in Chapter XI of their text.

If the transverse extension and relative displacement are negligibly small i.e. $E \to \infty$ and $w_3 \to 0$, then (21) is not needed and (49) reduces to

$$\lambda \tilde{p}^{\lambda}|_{\lambda} - \frac{(1+c)}{2}\bar{s}^{\lambda}|_{\lambda} + \frac{2_{\underline{0}}\lambda_{\underline{0}}\mu\lambda}{1-\eta}\left[-\bar{\lambda}L\bar{w}_3|^{\alpha\beta}_{\alpha\beta} + \frac{1}{G}\bar{s}^{\alpha}|^{\beta}_{\alpha\beta}\right] = 0. \tag{50}$$

Equation (50) is identically satisfied if the functions $\bar{s}^{\beta}|_{\beta}$ and $\bar{w}_3$ are derived from an invariant χ as follows [14].

$$\bar{s}^{\beta}|_{\beta} = -\frac{4L\lambda\bar{\lambda}_{\underline{0}}\lambda_{\underline{0}}\mu}{(1-\eta)(1+c)}\chi|^{\alpha\beta}_{\alpha\beta} - \bar{\lambda}LG\tilde{P}|^{\alpha}_{\alpha} \tag{51}$$

$$\bar{w}_3 = \chi - \frac{4\lambda_{\underline{0}}\lambda_{\underline{0}}\mu}{(1-\eta)G(1+c)}\chi|^{\alpha}_{\alpha} - \tilde{P} \tag{52}$$

in which $\tilde{P}$ is a particular integral of

$$\tilde{P}|^{\alpha}_{\alpha} = -\frac{2\lambda}{\bar{\lambda}LG(1+c)}\tilde{p}^{\alpha}|_{\alpha}. \tag{53}$$

If the approximation (35a) is used to determine $\bar{B}_{\alpha\beta}$, if w_3 is neglected and if equations (39), (47), (51) and (52) are introduced into (20), then the result is

$$\chi|^{\alpha\beta\gamma}_{\alpha\beta\gamma} - A\chi|^{\alpha\beta}_{\alpha\beta} + \bar{P} + B(\varepsilon^{\alpha\eta}\varepsilon^{\beta\gamma}\Phi|_{\gamma\eta} - \bar{F}^{\alpha\beta})(Lb_{\alpha\beta} + \chi|_{\alpha\beta} - C\chi|^{\mu}_{\mu\alpha\beta} - \tilde{P}|_{\alpha\beta}) = 0. \tag{54}$$

The constants, A, B, C, and the loading function $\bar{P}$ are given under the heading "*Notations*".

The right side of (45) can be expressed in terms of χ by means of (52), (35a) and (42). The problem of gross deflections ($w_3 = 0$) of shallow isotropic shells is thereby reduced to the determination of Φ and χ by the simultaneous solution of (54) and (45) without the underlined term. These differ from the corresponding equations of plates [1] only in the term $-BL\bar{F}^{\alpha\beta}b_{\alpha\beta}$ of (54).

SUMMARY

The preceding equations provide a basis for the analysis of thin sandwich shells with dissimilar facings. The basic equations are the equilibrium conditions (18)–(21), the constitutive equations (3) and (4) which describe the behavior of the core, the constitutive equations (22)–(25) together with the kinematic relations between the strains $\bar{\gamma}_{\alpha\beta}$, $\tilde{\gamma}_{\alpha\beta}$, changes-of-curvature $\bar{\kappa}_{\alpha\beta}$, $\kappa_{\alpha\beta}$, and the displacements $\bar{w}_i$ and w_i and the boundary conditions implied by (31) with (28) and (32). These equations are derived under the assumptions that the materials are linearly elastic, the core has negligible resistance to membrane forces, i.e. $\tau^{\alpha\beta} = 0$, the facings are thin enough to justify the Kirchhoff–Love approximation and the rotation about a normal is small of the same order-of-magnitude as the surface strain, i.e. $O(\omega_{\alpha\beta}) \approx O(\gamma_{\alpha\beta})$. The differential equations (18)–(21) and boundary conditions can be expressed in terms of the variables $\bar{w}_i$, σ^{33} and $\bar{s}^{\alpha}$ (gross displacement, mean transverse normal stress and transverse shear resultant, respectively), as discussed before.

Usually the transverse extensional stiffness is relatively large ($\lambda^2 G^{\alpha}/E \ll 1$) so that the second term of (4b) can be neglected. Then equation (4b) can be used to eliminate the variable $\bar{s}^{\alpha}$ in favor of w^{α}. Often the extensional modulus E is great enough that transverse extension is negligible. In this case, moderate deflections can be treated under the assumption that $w_3 = 0$.

With a few approximations the basic theory can be modified so that the equilibrium equations (18) and displacements $\bar{w}_\alpha$ can be eliminated in favor of a compatibility condition (44) or (45) and a stress function Φ. This also requires that the constitutive equation (46) replace (23) with (16b), and that (47) and (48) replace (24) and (25). These modifications rest on the assumptions that the facings are relatively thin, i.e. ${}_n\lambda \ll \lambda$, any moderate rotations are confined to shallow regions and the curvatures are constant, i.e. $b^\alpha_\beta|_\gamma = 0$ or negligibly small. With these approximations, the remaining equations-of-motion (19), (20), (21) and the compatibility condition (44) form a system of five differential equations in the variables Φ, $\bar{w}_3$, w_3, $\bar{s}^1$ and $\bar{s}^2$. As before, w_3 can often be neglected and $\bar{s}^\alpha$ can be eliminated in favor of w_α.

The theory for shallow isotropic shells can be expressed by two differential equations, (45) and (54), containing the invariants Φ and χ of (39), (51), and (52). It is understood that these are applicable to non-shallow shells when the deflection is localized or forms a pattern of short waves.

The theory of plates [1] is contained in the foregoing equations. Since the Donnell-type approximations are not needed to formulate the plate equations, it is evident that the precision of the shell theory improves with shallowness.

When the facings are geometrically similar the foregoing theory is equivalent to the previous formulation [2]. The equations have a different form because the present equations are expressed in terms of stress resultants which act at the middle surfaces of the facings. The choice of variables is that which provides the simplest formulation in each case.

If transverse extension of the core is negligible, the foregoing theory is equivalent to that given by Grigolyuk [15] for shells with similar facings.

Fulton [16] has employed the counterpart of equations (45), (49) and (50) to treat the buckling of a shallow cylindrical shell under a uniform axial thrust. His equations are derived by the principle of stationary potential energy with a kinematic variable replacing the invariant $\bar{s}^\alpha|_\alpha$ of (50); the tensorial counterpart of his kinematic variable is

$$\varphi = \left(2w^\gamma|_\gamma - \frac{{}_0\lambda + {}_1\lambda}{2}\bar{w}_3|^\gamma_\gamma\right).$$

If the nonlinear terms are suppressed, the foregoing theory is equivalent to that of Reissner [17] and, if transverse extensibility is negligible, the present equations reduce to the equations for shallow shells given by Stein and Mayers [18].

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Résumé—Ce rapport présente les équations differentielles et les conditions de limite pour des couches sandwiches à déflexions modérément larges. La théorie comprend la résistance de courbure des surfaces de portée, extension transversale et déformation de cisaillement de l'écorce. Les approximations et les simplifications sont décrites.

Zusammenfassung—Die Arbeit behandelt die Differentialgleichungen und Randbedingungen für Verbundschalen (Sandwich Shells) mit mässig grossen Winkelverdrehungen. Die Theorie berücksichtigt die Biegesteifigkeit der Aussenschichten, die Querdehnung und die Schubverformung des Kerns. Annäherungen und Vereinfachungen werden aufgeziegt.

Абстракт—Эта статья предлагает дифференциальные уравнения и граничные условия для прослойных оболочек с умеренно большими вращениями. Теория включает сопротивление изгибанию облицовок, поперечное расширение и дéформацию сдвига сердцевины. Описываются приближения и упрощения.